

ORBITAL STABILITY OF PERIODIC TRAVELING-WAVE SOLUTIONS FOR A DISPERSIVE EQUATION

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ABSTRACT. In this paper we establish the orbital stability of periodic traveling waves for a general class of dispersive equations. We use the Implicit Function Theorem to guarantee the existence of smooth solutions depending of the corresponding wave speed. Essentially, our method establishes that if the linearized operator has only one negative eigenvalue which is simple and zero is a simple eigenvalue the orbital stability is determined provided that a convenient condition about the average of the wave is satisfied. We use our approach to prove the orbital stability of periodic dnoidal waves associated with the Kawahara equation.

1. INTRODUCTION.

The existence of solutions that maintains its shape while it travels at constant speed is one of the most fascinating phenomena determined by dispersive equations. These special solutions (in general, called traveling waves) arise because of the perfect balance between the nonlinear and dispersive effects in the medium. In current literature, the existence of these solutions appear in several applications as fluid dynamics, nonlinear optics, hydrodynamic and many other fields. Thus, it is important to establish a qualitative study of the dynamic related to these special solutions.

The goal in this paper is to present sufficient conditions for the orbital stability of periodic traveling wave solutions related to the following general dispersive model,

$$u_t + uu_x - (\mathcal{M}u)_x = 0, \quad (1.1)$$

where $u : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a real L -periodic function and \mathcal{M} is a differential or pseudo-differential operator in periodic setting and it is defined as a Fourier multiplier operator by

$$\widehat{\mathcal{M}g}(\kappa) = \theta(\kappa)\widehat{g}(\kappa), \quad \kappa \in \mathbb{Z}, \quad (1.2)$$

where the symbol θ of \mathcal{M} is assumed to be a measurable, locally bounded function on \mathbb{R} , satisfying

$$A_1|\kappa|^{m_2} \leq \theta(\kappa) \leq A_2|\kappa|^{m_2}, \quad m_2 > 0, \quad (1.3)$$

for all $\kappa \in \mathbb{Z}$ and for some $A_i > 0$, $i = 1, 2$. Hypothesis (1.3) is necessary to study qualitative aspects of the model (1.1) (for instance, global well-posedness and stability) in the respective energy space associated, namely, $H_{per}^{\frac{m_2}{2}}([0, L])$. Now, since

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$\theta(0) = 0$ one has that \mathcal{M} satisfies

$$\mathcal{M}(a + u) = \mathcal{M}u \quad \text{and} \quad \int_0^L (\mathcal{M}u) dx = 0, \quad \forall a \in \mathbb{R}. \quad (1.4)$$

In equation (1.1), we consider traveling wave solutions of the form $u(x, t) = \psi(x - \omega t)$, where $\omega \in I \subset \mathbb{R}$ and $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function. So, if we substitute this form into (1.1), we obtain after integration

$$-\omega \psi_{(\omega, A)} + \frac{1}{2} \psi_{(\omega, A)}^2 - \mathcal{M} \psi_{(\omega, A)} + A = 0, \quad (1.5)$$

where A is a constant of integration not necessarily zero. A crucial role in our stability analysis is given by the symmetries of the model (1.1) in \mathbb{R} , namely,

- (1) *translation invariance*: $u(x, t) \rightarrow u(x + y, t)$, $y \in \mathbb{R}$;
- (2) *Galilean invariance*: $u(x, t) \rightarrow a + u(x, t)$, $a \in \mathbb{R}$.

So, if one considers the first condition in (1.4) and the Galilean invariance, we may assume $A \equiv 0$ in (1.5) for a specific value of parameter a . In addition, the Galilean invariance can be also used to construct positive, negative and sign-changed periodic solutions by taking a convenient value of a .

Particular cases of the operator \mathcal{M} and the respective result of orbital stability of periodic waves have been obtained by an extensive number of contributors. For instance, if one considers $\mathcal{M} = -\partial_x^2$ (the Korteweg-de Vries equation) we can cite [2] [3], [11], [14], and for $\mathcal{M} = \mathcal{H}\partial_x$ (the Benjamin-Ono equation), where \mathcal{H} indicates the Hilbert transform in periodic context, the first result of orbital stability of periodic waves was treated in [2]. When \mathcal{M} represents a fractionary derivative as $\mathcal{M} = (\sqrt{-\partial_x^2})^\alpha$, $0 < \alpha \leq 2$, in the Fourier sense (which includes the cases $\mathcal{M} = -\partial_x^2$ and $\mathcal{M} = \mathcal{H}\partial_x$), we have the work [13] where the authors assumed the existence of minimizers for the energy functional associated and proving the stability of periodic waves provided the number of negatives eigenvalues is one or two (to obtain the spectral property, they have used the approach in [9]).

Next, we shall give a brief outline of our work. In fact, let us consider the linearized operator around the wave $\psi_{(\omega, A)}$

$$\mathcal{L} = \mathcal{M} + \omega - \psi_{(\omega, A)}. \quad (1.6)$$

Operator \mathcal{L} in (1.6) is a closed, unbounded, self-adjoint operator on $L_{per}^2([0, L_0])$ whose spectrum consists in an enumerable (infinite) set of eigenvalues. Thus, by assuming that $\mathcal{L}_0 := \mathcal{L}|_{(\omega, A)=(\omega_0, A_0)}$ has only one negative eigenvalue which is simple and zero is a simple eigenvalue whose associated eigenfunction is $\frac{d}{dx}\psi$ (as required in [4], [6], [10] and [20]), we are able to establish the orbital stability of ψ provided that the average of the wave satisfies $\frac{1}{L_0} \int_0^{L_0} \psi(x) dx > \omega_0$. Our approach will be based on a combination of techniques determined by [6], [10], [14] and [20] where the construction of a smooth surface

$$(\omega, A) \in \mathcal{O} \mapsto \psi_{(\omega, A)} \in H_{per}^n([0, L_0]), \quad n \in \mathbb{N},$$

of even periodic waves which solves equation (1.5) is relevant in our analysis. Thus, in order to summarize our main assumption, we highlight it as follows

(H) Let $(\omega_0, A_0) \in \mathbb{R}_+ \setminus \{0\} \times \mathbb{R}$ be fixed. Suppose that $\psi := \psi_{(\omega_0, A_0)} \in C_{per}^\infty([0, L_0])$ is a positive even periodic traveling wave solution for the equation (1.5) with fixed period $L_0 > 0$. Moreover, the self-adjoint operator \mathcal{L}_0 has only one negative eigenvalue which is simple and zero is a simple eigenvalue whose eigenfunction is $\frac{d}{dx}\psi$.

As an application of our work, we present the result of orbital stability of periodic traveling waves for the Kawahara equation

$$u_t + uu_x + u_{xxx} - u_{xxxxx} = 0, \quad (1.7)$$

that is, $\mathcal{M} = \partial_x^4 - \partial_x^2$ in equation (1.1). The existence of explicit solutions is determined by using exhaustive numerical computations. In [19], the authors put forward an explicit periodic wave having a *dnoidal* profile as

$$\begin{aligned} \psi(x) = & a + b \left(\operatorname{dn}^2 \left(\frac{2K}{L}x, k \right) - \frac{E}{K} \right) \\ & + d \left(\operatorname{dn}^4 \left(\frac{2K}{L}x, k \right) - (2 - k^2) \frac{2E}{3K} + \frac{1-k^2}{3} \right), \end{aligned} \quad (1.8)$$

where *dn* is the Jacobi elliptic function called dnoidal, $k \in (0, 1)$ is the modulus, $K = K(k)$ indicates the complete integral elliptic of first kind and parameters a , b and d depend smoothly on the modulus $k \in (0, 1)$. Regarding the stability, in [12] the authors showed the linear stability of periodic waves (that is, the spectrum of the linearization about these waves is contained in the imaginary axis) related to the equation (1.7). They established the periodic travelling waves with speed ω are spectrally stable provided that the amplitude a of the wave satisfies $a = o(|\omega|^{5/4})$. In [7] it was determined a local proof for the orbital stability of periodic waves having the form (1.8) by using the arguments in [1]. Our goal is to determine a more complete scenario for the stability of periodic waves.

Our paper is organized as follows. Section 2 is devoted to present the stability of periodic waves associated with the general equation (1.1). In Section 3 we present the application of the results in previous section.

2. STABILITY OF PERIODIC WAVES

Before starting, we need to guarantee the existence of a smooth surface of periodic waves having fixed period. We see that assumption (H) is sufficient for our purpose.

Theorem 2.1. *Let us suppose that assumption (H) holds. There is a smooth surface of positive even periodic solutions for (1.5) and an open subset $\mathcal{O} \subset \mathbb{R}_+ \setminus \{0\} \times \mathbb{R}$, containing (ω_0, A_0) , such that*

$$(\omega, A) \in \mathcal{O} \mapsto \psi_{(\omega, A)} \in H_{per, e}^n([0, L_0]), \quad n \in \mathbb{N},$$

all of them with the same minimal period $L_0 > 0$.

Proof. We define for $s \geq 0$, $X_e^s = \{f \in H_{per}^s([0, L]) : f \text{ is even}\}$. Let $\Pi : R_+ \setminus \{0\} \times \mathbb{R} \times X_e^{m_2} \rightarrow X_e^0$ be the map defined by

$$\Pi(\omega, A, \psi) = \mathcal{M}\psi + \omega\psi - \frac{1}{2}\psi^2 + A. \quad (2.1)$$

Function Π is smooth in all variables and from assumption (H) one has $\Upsilon(\omega_0, A_0, \psi) = 0$. Next, the Fréchet derivative associated with the function Π with respect to ψ evaluated at the point (ω_0, A_0, ψ) becomes an operator \mathcal{G} given by

$$\mathcal{G} = \mathcal{M} + \omega_0 - \psi \quad (2.2)$$

Now, let us consider $f \in \ker(\mathcal{G})$, where \mathcal{G} is defined on X_e^0 with domain $D(\mathcal{G}) = X_e^{m_2}$. So, we have

$$\mathcal{M}f + \omega_0 f - \psi f = 0$$

Then $\frac{d}{dx}\psi$ is an eigenfunction of the operator $\mathcal{G} := \mathcal{M} + \omega_0 - \psi$ (as an operator defined in X^0 with domain X^{m_2}) whose eigenvalue is $\lambda = 0$. Moreover, since $\frac{d}{dx}\psi$ is odd and it does not belong to $X_e^{m_2}$, we see that \mathcal{G} is one to one. Now, let us prove that, with domain $X_e^{m_2}$, \mathcal{G} is also surjective. Indeed, \mathcal{G} is clearly a self-adjoint operator. Thus $\sigma(\mathcal{G}) = \sigma_{disc}(\mathcal{G}) \cup \sigma_{ess}(\mathcal{G})$. Since $X_e^{m_2}$ is compactly embedded in X_e^0 , the operator \mathcal{G} has compact resolvent. Consequently, $\sigma_{ess}(\mathcal{G}) = \emptyset$ and $\sigma(\mathcal{G}) = \sigma_{disc}(\mathcal{G})$ consists of isolated eigenvalues with finite algebraic multiplicities (see [16]). Now, since \mathcal{G} is one-to-one, it follows that 0 is not an eigenvalue of \mathcal{G} , and so it does not belong to $\sigma(\mathcal{G})$. This means that $0 \in \rho(\mathcal{G})$, where $\rho(\mathcal{G})$ denotes the resolvent set of \mathcal{G} , and so, by definition, \mathcal{G} is surjective. The arguments above imply that \mathcal{G}^{-1} exists and, moreover, is a bounded linear operator. Consequently, since Π and Π_ψ are clearly smooth maps on their domains, from the Implicit Function Theorem we establish the results enunciated above. \square

Next result establishes the behaviour of the first eigenvalues associated with the linearized operator \mathcal{L}_0 in (1.6).

Proposition 2.1. *Suppose that assumption (H) holds and let $\psi_{(\omega, A)}$ be the periodic traveling wave solution obtained in Theorem 2.1. Operator $\mathcal{L} = \mathcal{M} + \omega - \psi_{(\omega, A)}$, $(\omega, A) \in \mathcal{O}$, has only one negative eigenvalue which is simple and zero is a simple eigenvalue whose eigenfunction is $\frac{d}{dx}\psi_{(\omega, A)}$.*

Proof. Indeed, from Theorem 2.1, let us consider $(\omega, A) \in \mathcal{O}$. The family of self-adjoint operators $\mathcal{L} = \mathcal{M} + \omega - \psi_{(\omega, A)}$ is defined on $L_{per}^2([0, L])$ with domain $D(\mathcal{L}) = H_{per}^{m_2}([0, L])$. Thus, if we consider the metric gap, $\widehat{\delta}(T, S)$, between the closed operators T and S (see Chap. IV in [16]) we have from Theorem 2.17 and Theorem 2.14 in Chap. IV of [16],

$$\begin{aligned} \widehat{\delta}(\mathcal{L}_{(\omega_0, A_0)}, \mathcal{L}) &\leq 2(1 + \|\psi_{(\omega, A)}\|_{L^\infty}^2) \widehat{\delta}(\mathcal{L}_{(\omega_0, A_0)} + \psi_{(\omega, A)}, \mathcal{M} + \omega) \\ &\leq 2(1 + \|\psi_{(\omega, A)}\|_{L^\infty}^2) [\|\omega_0 - \omega\| + \|\psi_{(\omega, A)} - \psi_{(\omega_0, A_0)}\|_{L^\infty}]. \end{aligned} \quad (2.3)$$

Therefore we obtain $\widehat{\delta}(\mathcal{L}_{(\omega_0, A_0)}, \mathcal{L}) \rightarrow 0$ as $(\omega, A) \rightarrow (\omega_0, A_0)$, and so from [16, Theorem 3.16, Chap. IV]) the isolated eigenvalues of $\mathcal{L}_{(\omega_0, A_0)}$ are stable. Hence, for $(\omega, A) \in \mathcal{O}$, we obtain that \mathcal{L} has the same spectral properties of $\mathcal{L}_{(\omega_0, A_0)}$. \square

Next, we present our stability result by adapting the arguments in [5], [10], [14] and [20] (for details, we also refer the reader to see [8]). So, in what follows, we assume that the model in (1.1) possesses a convenient global well-posedness result in the space $H_{per}^s([0, L_0])$, for $s \geq \frac{m_2}{2}$. In addition, we need to suppose the existence of the following conserved quantities

$$E(u) = \frac{1}{2} \int_0^{L_0} (\mathcal{M}^{1/2} u)^2 - \frac{1}{3} u^3 dx, \quad (2.4)$$

$$F(u) = \frac{1}{2} \int_0^{L_0} u^2 dx, \quad (2.5)$$

and

$$M(u) = \int_0^{L_0} u dx, \quad (2.6)$$

where in the quantity (2.4) we are using that operator \mathcal{M} is *m-accretive* (see [16, pg. 281]). This fact allows us to conclude the existence of a self-adjoint linear operator $\mathcal{M}^{1/2}$ such that $(\mathcal{M}^{1/2})^2 = \mathcal{M}$.

Assume that assumption (H) holds. From Theorem 2.1 we are enabled to consider

$$\eta := \frac{\partial}{\partial \omega} \psi_{(\omega, A)} \Big|_{(\omega, A) = (\omega_0, A_0)}, \quad \beta := \frac{\partial}{\partial A} \psi_{(\omega, A)} \Big|_{(\omega, A) = (\omega_0, A_0)}.$$

Define

$$\begin{aligned} M(\psi) &= \int_0^{L_0} \psi_{(\omega, A)}(x) dx \Big|_{(\omega, A) = (\omega_0, A_0)}, & F(\psi) &= \frac{1}{2} \int_0^{L_0} \psi_{(\omega, A)}^2(x) dx \Big|_{(\omega, A) = (\omega_0, A_0)}, \\ M_\omega(\psi) &= \frac{\partial}{\partial \omega} \int_0^{L_0} \psi_{(\omega, A)}(x) dx \Big|_{(\omega, A) = (\omega_0, A_0)}, & M_A(\psi) &= \frac{\partial}{\partial A} \int_0^{L_0} \psi_{(\omega, A)}(x) dx \Big|_{(\omega, A) = (\omega_0, A_0)}, \end{aligned}$$

and

$$F_\omega(\psi) = \frac{1}{2} \frac{\partial}{\partial \omega} \int_0^{L_0} \psi_{(\omega, A)}^2(x) dx \Big|_{(\omega, A) = (\omega_0, A_0)}, \quad F_A(\psi) = \frac{1}{2} \frac{\partial}{\partial A} \int_0^{L_0} \psi_{(\omega, A)}^2(x) dx \Big|_{(\omega, A) = (\omega_0, A_0)}.$$

Now, we need some preliminaries notations. First, the norm and inner product in $L_{per}^2([0, L_0])$ will be denoted by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$. Now, let ρ be the semi-distance defined on the energy space $X^{\frac{m_2}{2}} = H_{per}^{\frac{m_2}{2}}([0, L_0])$ as

$$\rho(u, \psi) = \inf_{y \in \mathbb{R}} \|u(\cdot + y) - \psi\|_{X^{\frac{m_2}{2}}}. \quad (2.7)$$

For a given $\varepsilon > 0$, we define the ε -neighborhood of the orbit O_ψ as

$$U_\varepsilon := \{u \in X^{\frac{m_2}{2}}; \rho(u, \psi) < \varepsilon\}. \quad (2.8)$$

We also introduce the smooth manifolds

$$\Sigma_0 = \{u \in X^{\frac{m_2}{2}}; F(u) = F(\psi), M(u) = M(\psi)\}, \quad (2.9)$$

and

$$\Upsilon_0 = \{u \in X^{\frac{m_2}{2}}; \langle \psi, u \rangle = \langle 1, u \rangle = 0\}. \quad (2.10)$$

Our notion of orbital stability is finally presented.

Definition 2.1. We say that ψ is orbitally stable with respect to (1.1) if, for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|u_0 - \psi\|_{X^{\frac{m_2}{2}}} < \delta$ and $u(t)$ is the solution of (1.1) with $u(0) = u_0$, then

$$\rho(u(t), \psi) < \varepsilon, \quad \text{for all } t \in \mathbb{R}.$$

The next result states that under a suitable restriction, the operator \mathcal{L} is strictly positive.

Proposition 2.1. Suppose that assumption (H) holds. Assume that there is $\Phi \in X^{m_2}$ such that $\langle \mathcal{L}_0 \Phi, \varphi \rangle = 0$, for all $\varphi \in \Upsilon_0$, and

$$\mathcal{I} := \langle \mathcal{L}_0 \Phi, \Phi \rangle < 0 \quad (2.11)$$

Then, there is a constant $c > 0$ such that

$$\langle \mathcal{L}_0 v, v \rangle \geq c \|v\|_{X^{\frac{m_2}{2}}}^2,$$

for all $v \in \Upsilon_0$ such that $\langle v, \psi' \rangle = 0$.

Proof. We shall give only a sketch of the proof. From assumption (H) one has

$$L_{per}^2([0, L_0]) = [\chi] \oplus [\psi'] \oplus P, \quad (2.12)$$

where χ satisfies $\|\chi\| = 1$ and $\mathcal{L}_0 \chi = -\lambda_0^2 \chi$, $\lambda_0 \neq 0$. By using the arguments in [16, page 278], we obtain that

$$\langle \mathcal{L}_0 p, p \rangle \geq c_1 \|p\|^2, \quad \text{for all } p \in D(\mathcal{L}) \cap P,$$

where c_1 is a positive constant.

Next, from (2.32), we write

$$\Phi = a_0 \chi + b_0 \psi' + p_0, \quad a_0, b_0 \in \mathbb{R},$$

where $p_0 \in D(\mathcal{L}_0) \cap P$. Now, since $\psi' \in \ker(\mathcal{L}_0)$, $\mathcal{L}_0 \chi = -\lambda_0^2 \chi$, and $\mathcal{I} < 0$, we obtain

$$\langle \mathcal{L}_0 p_0, p_0 \rangle = \langle \mathcal{L}_0 (\Phi - a_0 \chi - b_0 \psi'), \Phi - a_0 \chi - b_0 \psi' \rangle = \langle \mathcal{L}_0 \Phi, \Phi \rangle + a_0^2 \lambda_0^2 < a_0^2 \lambda_0^2. \quad (2.13)$$

Taking $\varphi \in \Upsilon_0$ such that $\|\varphi\| = 1$ and $\langle \varphi, \psi' \rangle = 0$, we can write $\varphi = a_1 \chi + p_1$, where $p_1 \in X^{\frac{m_2}{2}} \cap P$. Thus,

$$0 = \langle \mathcal{L}_0 \Phi, \varphi \rangle = \langle -a_0 \lambda_0^2 \chi + \mathcal{L}_0 p_0, a_1 \chi + p_1 \rangle = -a_0 a_1 \lambda_0^2 + \langle \mathcal{L}_0 p_0, p_1 \rangle. \quad (2.14)$$

The rest of the proof runs as in [5, Lemma 5.1] (see also [14, Lemma 4.4]). \square

Proposition 2.1 is useful to establish the following result.

Proposition 2.2. Let E be the conserved quantity defined in (2.4). Under the assumptions of Proposition 2.1 there are $\alpha > 0$ and $C = C(\alpha) > 0$ such that

$$E(u) - E(\psi) \geq C \rho(u, \psi)^2,$$

for all $u \in U_\alpha \cap \Sigma_0$.

Proof. The proof can be found in [14, Lemma 4.6]. So, we omit the details. \square

Finally, we present sufficient conditions for the stability.

Theorem 2.3. *Assume that assumption (H) holds and let us suppose that*

$$\mathcal{D} := \begin{bmatrix} F_A(\psi) & M_A(\psi) \\ F_\omega(\psi) & M_\omega(\psi) \end{bmatrix}$$

is invertible. If there is $\Phi \in X^{m_2}$ such that $\langle \mathcal{L}_0 \Phi, \varphi \rangle = 0$, for all $\varphi \in \Upsilon_0$, and $\mathcal{I} = \langle \mathcal{L}_0 \Phi, \Phi \rangle < 0$, then ψ is orbitally stable in $X^{\frac{m_2}{2}}$ by the periodic flow of (1.1).

Proof. Let $\alpha > 0$ be the constant such that Proposition 2.2 holds. Since E is continuous at ψ , for a given $\varepsilon > 0$, there exists $\delta \in (0, \alpha)$ such that if $\|u_0 - \psi\|_{X^{\frac{m_2}{2}}} < \delta$ one has

$$E(u_0) - E(\psi) < M\varepsilon^2, \quad (2.15)$$

where $M > 0$ is the constant in Proposition 2.2. We need to divide our proof into two cases.

First case. $u_0 \in \Sigma_0$. Since F and M are conserved quantities, if $u_0 \in \Sigma_0$ one has that $u(t) \in \Sigma_0$, for all $t \geq 0$. The time continuity of the function $\rho(u(t), \psi)$ allows to choose $T > 0$ such that

$$\rho(u(t), \psi) < \alpha, \quad \text{for all } t \in [0, T]. \quad (2.16)$$

Thus, one obtains $u(t) \in U_\alpha$, for all $t \in [0, T]$. Combining Proposition 2.2 and (2.15), we have

$$\rho(u(t), \psi) < \varepsilon, \quad \text{for all } t \in [0, T]. \quad (2.17)$$

Next, we prove that $\rho(u(t), \psi) < \alpha$, for all $t \in [0, +\infty)$, from which one concludes the orbital stability restricted to perturbations in the manifold Σ_0 . Indeed, let $T_1 > 0$ be the supremum of the values of $T > 0$ for which (2.16) holds. To obtain a contradiction, suppose that $T_1 < +\infty$. By choosing $\varepsilon < \frac{\alpha}{2}$ we obtain, from (2.17),

$$\rho(u(t), \psi) < \frac{\alpha}{2}, \quad \text{for all } t \in [0, T_1].$$

Since $t \in (0, +\infty) \mapsto \rho(u(t), \psi)$ is continuous, there is $T_0 > 0$ such that $\rho(u(t), \psi) < \frac{3}{4}\alpha < \alpha$, for $t \in [0, T_1 + T_0]$, contradicting the maximality of T_1 . Therefore, $T_1 = +\infty$ and the theorem is established if $u_0 \in \Sigma_0$.

Second case. $u_0 \notin \Sigma_0$. In this case, since $\det(\mathcal{D}) \neq 0$, we claim that there is $(\omega_1, A_1) \in \mathcal{O}$, such that $F(\psi_{(\omega_1, A_1)}) = F(u_0)$ and $M(\psi_{(\omega_1, A_1)}) = M(u_0)$.

In fact, since M and F are smooth, the Inverse Function Theorem implies the existence of $r_1, r_2 > 0$ such that the map

$$\begin{aligned} \Gamma : B_{r_1}(\omega_0, A_0) &\longrightarrow B_{r_2}(M(\psi), F(\psi)) \\ (\omega, A) &\longmapsto (M(\psi_{(\omega, A)}), F(\psi_{(\omega, A)})) \end{aligned}$$

is a smooth diffeomorphism. Here, $B_r((x, y))$ denotes the open ball in \mathbb{R}^2 centered in (x, y) with radius $r > 0$. The continuity of the functionals M and V gives (if necessary we can take a smaller $\delta > 0$)

$$|M(u_0) - M(\psi)| < \frac{r_2}{\sqrt{2}} \quad \text{and} \quad |F(u_0) - F(\psi)| < \frac{r_2}{\sqrt{2}},$$

that is, $(M(u_0), F(u_0)) \in B_{r_2}(M(\psi), F(\psi))$. Since Γ is a diffeomorphism, there is a unique $(\omega_1, A_1) \in B_{r_1}(\omega_0, A_0)$ such that $(M(u_0), F(u_0)) = (M(\psi_{(\omega_1, A_1)}), F(\psi_{(\omega_1, A_1)}))$. The claim is thus proved.

The remainder of the proof follows from the smoothness of the periodic wave with respect to the parameters, the fact that the period does not change whether $(\omega, A) \in \mathcal{O}$ and the triangle inequality. \square

Theorem 2.3 establishes the orbital stability of ψ provided $\det(\mathcal{D}) \neq 0$ and $\mathcal{I} < 0$. The next proposition gives a sufficient condition to show that $\mathcal{I} < 0$.

Proposition 2.4. *Let $P : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the function defined as*

$$P(x, y) = x^2 F_\omega(\psi) + xy(F_A(\psi) + M_\omega(\psi)) + y^2 M_A(\psi).$$

Assume that there is $(x_0, y_0) \in \mathbb{R}^2$ such that $P(x_0, y_0) > 0$. Then there is $\Phi \in X^{m_2}$ such that $\langle \mathcal{L}_0 \Phi, \varphi \rangle = 0$, for all $\varphi \in \Upsilon_0$, and

$$\mathcal{I} = \langle \mathcal{L}_0 \Phi, \Phi \rangle < 0.$$

Proof. It suffices to define $\Phi := x_0 \eta + y_0 \beta$. Indeed, since $\mathcal{L}_0 \beta = -1$ and $\mathcal{L}_0 \eta = -\psi$, it is clear that $\langle \mathcal{L}_0 \Phi, \varphi \rangle = 0$, for all $\varphi \in \Upsilon_0$, and

$$\begin{aligned} \langle \mathcal{L}_0 \Phi, \Phi \rangle &= \langle -x_0 \psi - y_0, x_0 \eta + y_0 \beta \rangle \\ &= -(x_0^2 F_\omega(\psi) + x_0 y_0 F_A(\psi) + x_0 y_0 M_\omega(\psi) + y_0^2 M_A(\psi)) \\ &= -P(x_0, y_0). \end{aligned}$$

The proof is thus completed. \square

Corollary 2.1. *Suppose that assumption (H) occurs. If $\frac{M(\psi)}{L_0} > \omega_0 > 0$, there exists $(x_0, y_0) \in \mathbb{R}^2$ such that $P(x_0, y_0) > 0$.*

Proof. From assumption (H) one gets Theorem 2.1 and consequently, it is possible to derive equation (1.5) with respect to ω and A to obtain, respectively

$$\mathcal{M}\eta + \omega\eta + \psi - \psi\eta = 0, \quad (2.18)$$

and,

$$\mathcal{M}\beta + \omega\beta - \psi\beta + 1 = 0. \quad (2.19)$$

Next, integrating equations (2.18) and (2.19) over $[0, L_0]$ we deduce, respectively

$$F_\omega(\psi) = \omega M_\omega(\psi) + M(\psi), \quad (2.20)$$

and,

$$F_A(\psi) = \omega M_A(\psi) + L_0. \quad (2.21)$$

On the other hand, since $1 \in D(\mathcal{L})$ we have from (1.6) that $\mathcal{L}1 = \omega - \psi_{(\omega, A)}$. The fact that $1, \psi_{(\omega, A)} \in [\psi'_{(\omega, A)}]^\perp$, for all $(\omega, A) \in \mathcal{O}$, enables us to obtain $1 = \omega \mathcal{L}^{-1}1 - \mathcal{L}^{-1}\psi_{(\omega, A)}$, for all $(\omega, A) \in \mathcal{O}$. Therefore, from (1.6), (2.18) and (2.19) we conclude

$$L_0 = -\omega_0 M_A(\psi) + M_\omega(\psi). \quad (2.22)$$

So, collecting the results in (2.20), (2.21) and (2.22) we have

$$\begin{aligned} P(x_0, y_0) &= x_0^2 F_\omega(\psi) + x_0 y_0 F_A(\psi) + x_0 y_0 M_\omega(\psi) + y_0^2 M_A(\psi) \\ &= \left(x_0^2 \omega_0 + 2x_0 y_0 + \frac{y_0^2}{\omega_0} \right) M_\omega(\psi) + x_0^2 M(\psi) - \frac{y_0^2 L_0}{\omega_0} \end{aligned} \quad (2.23)$$

Choosing $y_0 \neq 0$ and $x_0 = -\frac{y_0}{\omega_0}$, one has from (2.23)

$$P(x_0, y_0) = \frac{y_0^2}{\omega_0^2} M(\psi) - \frac{y_0^2 L_0}{\omega_0} = \frac{y_0^2 L_0}{\omega_0^2} \left(\frac{M(\psi)}{L_0} - \omega_0 \right). \quad (2.24)$$

The fact that $\frac{M(\psi)}{L_0} > \omega_0$ enables us to finish the proof. \square

Corollary 2.2. *Suppose that assumption (H) occurs. Thus*

$$\det(\mathcal{D}) = \frac{L_0}{\omega_0} \left(\omega_0 - \frac{M(\psi)}{L_0} \right) M_\omega(\psi) + M(\psi) \frac{L_0}{\omega_0}.$$

In particular, if $\frac{M(\psi)}{L_0} > \omega_0 > 0$ and $M_\omega(\psi) < 0$ one has that $\det(\mathcal{D}) \neq 0$

Proof. In fact, from (2.20), (2.21) and (2.22) we have

$$\begin{aligned} \det(\mathcal{D}) &= F_A(\psi) M_\omega(\psi) - F_\omega(\psi) M_A(\psi) \\ &= \omega_0 M_A(\psi) M_\omega(\psi) + L_0 M_\omega(\psi) - \omega_0 M_\omega(\psi) M_A(\psi) - M(\psi) M_A(\psi) \\ &= L_0 M_\omega(\psi) - M(\psi) M_A(\psi) = L_0 M_\omega(\psi) + \frac{M(\psi) L_0}{\omega_0} - \frac{M_\omega(\psi) M(\psi)}{\omega_0} \\ &= \frac{L_0}{\omega_0} \left(\omega_0 - \frac{M(\psi)}{L_0} \right) M_\omega(\psi) + \frac{M(\psi) L_0}{\omega_0}. \end{aligned} \quad (2.25)$$

The proof is now completed. \square

Next, let us suppose that $\frac{M(\psi)}{L_0} > \omega_0 > 0$. From Proposition 2.4, Corollary 2.1 and Corollary 2.2, we deduce from Theorem 2.3 that the periodic wave ψ is orbitally stable in $X^{\frac{m_2}{2}}$ by the periodic flow of (1.5) provided that $M_\omega(\psi) < 0$. Therefore, we need to determine the behaviour of the stability if one considers $M_\omega(\psi) \geq 0$. This particular case is proved in a different way since we can not assure that $\det(\mathcal{D}) \neq 0$ in order to apply the arguments in Theorem 2.3. However, it is easy to see that from (2.20) one has $P(1, 0) = F_\omega(\psi) = \omega_0 M_\omega(\psi) + M(\psi) > 0$. This information about the positivity of $F_\omega(\psi)$ will be useful for our purpose.

In this case, we follow the arguments contained in [18]. In fact, let us consider

$$\mathcal{P}_{(\omega, A)} = E + \omega F + AM \quad (2.26)$$

and the perturbation

$$u(x + y, t) = \psi_{(\omega, A)}(x) + v(x, t), \quad (2.27)$$

where $y = y(t)$ is the minimum point of the function

$$\Gamma_t(s) = \int_0^{L_0} (\mathcal{M}^{1/2}(u(x + y, t) - \psi_{(\omega, A)})) ^2 dx + \omega \int_0^{L_0} (u(x + y, t) - \psi_{(\omega, A)}(x))^2 dx,$$

$y \in \mathbb{R}$, and function v satisfies the compatibility condition

$$\int_0^{L_0} \psi_{(\omega, A)}(x) \psi'_{(\omega, A)}(x) v(x, t) dx = 0, \quad (2.28)$$

for all $t \in \mathbb{R}$.

Thus, we obtain from (2.26) and (2.27) the following inequality

$$\begin{aligned} \Delta \mathcal{P}_{(\omega, A)} &:= \mathcal{P}_{(\omega, A)}(u) - \mathcal{P}_{(\omega, A)}(\psi_{(\omega, A)}) = \mathcal{P}_{(\omega, A)}(\psi_{(\omega, A)} + v) - \mathcal{P}_{(\omega, A)}(\psi_{(\omega, A)}) \\ &\geq \frac{1}{2} \langle \mathcal{L}v, v \rangle - C_0 \|v\|_{X^{\frac{m_2}{2}}}^3, \end{aligned} \quad (2.29)$$

where $C_0 \in \mathbb{R}$ is a positive constant which depend on the periodic wave $\psi_{(\omega, A)}$ and the constant of the Sobolev embeddings $X^{\frac{m_2}{2}} \hookrightarrow L_{per}^p([0, L_0])$, $p \geq 2$, integer.

Next, it is necessary to use the works due to [4] and [6], to establish convenient bounds for the term $\langle \mathcal{L}v, v \rangle$. First, we need a preliminary result.

Lemma 2.1. *Let $\psi_{(\omega, A)}$ be as in Theorem 2.1. Let \mathcal{L} be the self-adjoint operator defined in (1.6). We define*

$$-\infty < w := \min_{\phi} \{ \langle \mathcal{L}\phi, \phi \rangle; \|\phi\|_{L_{per}^2} = 1 \text{ and } \langle \phi, \psi_{(\omega, A)} \rangle = 0 \}.$$

Assuming that $\langle \chi_{(\omega, A)}, \psi_{(\omega, A)} \rangle \neq 0$ and $\psi_{(\omega, A)} \in [\ker(\mathcal{L})]^\perp$, where $\chi_{(\omega, A)}$ is the eigenfunction associated with the negative eigenvalue of \mathcal{L} . Then, if

$$\langle \mathcal{L}^{-1}\psi_{(\omega, A)}, \psi_{(\omega, A)} \rangle \leq 0, \quad (2.30)$$

it follows that $w \geq 0$.

Proof. See Lemma E.1 in [21]. □

Remark 2.1. *To obtain that $\langle \chi_{(\omega, A)}, \psi_{(\omega, A)} \rangle \neq 0$ it makes necessary to use that $\psi_{(\omega, A)}$ is positive and the eigenfunction χ , related to the first eigenfunction of \mathcal{L} , is one-signed (this last fact can be obtained by applying the well known Krein-Ruttman Theorem).*

Next, since

$$\langle \mathcal{L}^{-1}\psi_{(\omega, A)}, \psi_{(\omega, A)} \rangle = -\frac{1}{2} \frac{d}{d\omega} \int_0^{L_0} \psi_{(\omega, A)}^2(x) dx,$$

one has that (2.30) occurs at the point $(\omega_0, A_0) \in \mathcal{O}$ if, and only if

$$\frac{1}{2} \frac{d}{d\omega} \int_0^{L_0} \psi_{(\omega, A)}^2(x) dx \Big|_{(\omega, A) = (\omega_0, A_0)} = F_\omega(\psi) > 0. \quad (2.31)$$

Thus, by using that $M_\omega(\psi) > 0$ and $M(\psi) > 0$ that $\langle \mathcal{L}^{-1}\psi_{(\omega, A)}, \psi_{(\omega, A)} \rangle < 0$ for all $(\omega, A) \in \mathcal{O}$.

Lemma 2.1 jointly with (2.31) will be useful to establish next result.

Lemma 2.2. *Let $\psi_{(\omega, A)}$ be as in Theorem 2.1. Then, for $(\omega, A) \in \mathcal{O}$, we have*

$$(i) \inf \{ \langle \mathcal{L}f, f \rangle; \|f\| = 1, \langle f, \psi_{(\omega, A)} \rangle = 0 \} = 0.$$

$$(ii) \inf \{ \langle \mathcal{L}f, f \rangle; \|f\| = 1, \langle f, \psi_{(\omega, A)} \rangle = 0, \langle f, \psi_{(\omega, A)} \psi'_{(\omega, A)} \rangle = 0 \} > 0.$$

Proof. The proof follows from similar arguments as in [18, Lemma 4.2]. □

We now present the proof of the stability of periodic waves for the equation (1.5) in the case that $M_\omega(\psi) > 0$ and $\frac{M(\psi)}{L_0} > \omega_0$. Firstly, we estimate term $\langle \mathcal{L}v, v \rangle$ from below by assuming without loss of generality that $\|\psi_{(\omega,A)}\| = 1$. Let us define

$$v_\perp = v - v_\parallel, \quad \text{where} \quad v_\parallel = \langle v, \psi_{(\omega,A)} \rangle \psi_{(\omega,A)}. \quad (2.32)$$

From (2.28) and (2.32) one has

$$\langle v_\perp, \psi_{(\omega,A)} \psi'_{(\omega,A)} \rangle = \langle v, \psi_{(\omega,A)} \psi'_{(\omega,A)} \rangle - \langle v_\parallel, \psi_{(\omega,A)} \psi'_{(\omega,A)} \rangle = \langle v, \psi_{(\omega,A)} \rangle \langle \psi_{(\omega,A)}^2, \psi'_{(\omega,A)} \rangle = 0.$$

In addition, since $\langle v_\perp, \psi_{(\omega,A)} \rangle = 0$, Lemma 2.2 yields

$$\langle \mathcal{L}v_\perp, v_\perp \rangle \geq C_1 \|v_\perp\|^2, \quad (2.33)$$

for some $C_1 > 0$.

Assuming first that $\|u_0\| = \|\psi_{(\omega,A)}\| = 1$. Since F is a conserved quantity, we obtain $\|u(t)\|^2 = 1$ for all t . Hence, because (1.5) is invariant by translations, we obtain $\langle v, \psi_{(\omega,A)} \rangle \geq -C_2 \|v\|_{X^{\frac{m_2}{2}}}^4$. Thus, there are positive constants C_3 and C_4 such that

$$\langle \mathcal{L}v_\perp, v_\perp \rangle \geq C_3 \|v\|^2 - C_4 \|v\|_{X^{\frac{m_2}{2}}, \omega}^4, \quad (2.34)$$

where $\|f\|_{X^{\frac{m_2}{2}}, \omega}^2 := \int_0^{L_0} (\mathcal{M}^{1/2} f(x))^2 dx + \omega \int_0^{L_0} f(x)^2 dx$ is an equivalent norm in $X^{\frac{m_2}{2}}$. Next, from the Cauchy-Schwartz inequality,

$$\langle \mathcal{L}v_\parallel, v_\parallel \rangle \geq -C_5 \|v\|_{X^{\frac{m_2}{2}}, \omega}^3, \quad (2.35)$$

for some $C_5 > 0$. Therefore, (2.34) and (2.35), yield

$$\langle \mathcal{L}v, v \rangle \geq C_6 \|v\|_{X^{\frac{m_2}{2}}, \omega}^2 - C_7 \|v\|_{X^{\frac{m_2}{2}}, \omega}^3 - C_8 \|v\|_{X^{\frac{m_2}{2}}, \omega}^4, \quad (2.36)$$

where $C_i > 0$, $i = 6, 7, 8$. Finally, collecting results in (2.29) and (2.36) we have

$$\Delta \mathcal{P}_{(\omega,A)} \geq D_1 \|v\|_{X^{\frac{m_2}{2}}, \omega}^2 - D_2 \|v\|_{X^{\frac{m_2}{2}}, \omega}^3 - D_3 \|v\|_{X^{\frac{m_2}{2}}, \omega}^4, \quad (2.37)$$

for some $D_i > 0$, $i = 1, 2, 3$. The remainder of the proof can be established by using standard arguments. For details, we refer the reader to see [6] (see also [3] and [20]). This argument proves that the orbit generated by $\psi_{(\omega,A)}(x - ct)$ is stable relative to small perturbations which preserves the L_{per}^2 -norm of $\psi_{(\omega,A)}$. The general case (that for $\|u_0\| \neq \|\psi_{(\omega,A)}\|$) follows from the continuous dependence of the function $\psi_{(\omega,A)}$ with respect to the parameters (ω, A) jointly with the triangle inequality. \square

3. AN APPLICATION

This section is devoted to apply the arguments in Section 2 to conclude the orbital stability of periodic waves for the Kawahara equation (1.7). In reference [7], the authors have constructed a smooth curve $\omega \in I \mapsto \psi_\omega \in H_{per}^n([0, L_0])$, $n \in \mathbb{N}$, of L_0 -periodic waves and proving the orbital stability for specific values of $\omega \in I$ by using the arguments in [1]. The method established in [1] was an adaptation for the periodic case of the classical theory established in [10]. In our present approach, we prove the orbital stability without assuming the restrictions on the wave speed ω as

contained in [7].

Indeed, let us consider the ansatz (see [19])

$$\begin{aligned} \psi(x) = a &+ b \left(\operatorname{dn}^2 \left(\frac{2K}{L}x, k \right) - \frac{E}{K} \right) \\ &+ d \left(\operatorname{dn}^4 \left(\frac{2K}{L}x, k \right) - (2 - k^2) \frac{2E}{3K} + \frac{1 - k^2}{3} \right). \end{aligned} \quad (3.1)$$

Substituting this form into the equation

$$\psi'''' - \psi'' + \omega\psi - \frac{1}{2}\psi^2 + A = 0 \quad (3.2)$$

one has explicit periodic solutions provided that

$$\begin{aligned} a = \frac{1}{507L^4} &((-k^4 + k^2 + 1)302848K^4 + 14560L^2K^2(k^2 - 2) \\ &+ 43680L^2EK + L^4(-31 + 507\omega)), \end{aligned} \quad (3.3)$$

$$b = \frac{1120}{13L^4}((208k^2 - 416)K^2 + L^2)K^2 \quad \text{and} \quad d = \frac{26880K^4}{L^4}. \quad (3.4)$$

Furthermore, A is a complicated function which depends smoothly on the triple (k, L, ω) and it may be expressed by

$$A = f_1(k, L) + C\omega^2, \quad (3.5)$$

where $C \in \mathbb{R}$.

Moreover, we also need to consider a pair (k, L) which solves the following (implicit) nonlinear equation

$$\frac{89989120}{31}(k^2 - 2) \left(k^2 - \frac{1}{2} \right) (k^2 + 1)K^6 - \frac{908544}{31}L^2(k^4 - k^2 + 1)K^4 + L^6 = 0. \quad (3.6)$$

A standard application of the implicit function theorem gives us the existence of two open intervals $I \subset (0, +\infty)$ and $J \subset (0, 1)$ such that the function $k \in J \mapsto L(k) \in I$ is smooth. Therefore, for a fixed value of the modulus $k_0 \in (0, 1)$ one has a unique value $L_0 > 0$ such that ψ is a smooth L_0 -periodic solution related to the equation (3.2) as required in the first part of assumption (H) (important to mention that ω is a free parameter which does not depend on the pair (k, L)).

With this arguments in hands, we need to establish the spectral property associated with the linearized operator

$$\mathcal{L}_{(\omega_0, A_0)} = \partial_x^4 - \partial_x^2 + \omega_0 - \psi, \quad \omega_0 > 0. \quad (3.7)$$

Proposition 3.1. *Consider $L_0 > 0$ satisfying (3.6) and let $\omega_0 > 0$ be arbitrary but fixed. The operator $\mathcal{L}_{(\omega_0, A_0)}$ in (3.7) possesses exactly a unique negative eigenvalue which is simple, and zero is a simple eigenvalue with eigenfunction $\frac{d}{dx}\psi$.*

Proof. We prove the result by using Theorem 4.1 in [2]. First of all, we use the Galilean invariance associated to (3.2) in order to prove that the spectral property in (H) will be the same for all values of $\omega \in \mathbb{R}$ (the value of the integration constant A is irrelevant in our spectral analysis). Therefore, it suffices to prove the result

for a specific value of ω_0 . In fact, let $\alpha_0 \in \mathbb{R}$ be arbitrary but fixed. By defining $\tilde{\psi} = \alpha_0 + \psi$, where ψ is solution of (3.2), it follows that

$$(\omega_0 + \alpha_0)\tilde{\psi} - \frac{1}{2}\tilde{\psi}^2 - \tilde{\psi}'' + \tilde{\psi}'''' + \tilde{A}_0 = 0,$$

where $\tilde{A}_0 = A_0 - \omega_0\alpha_0 - \frac{\alpha_0^2}{2}$. Therefore, $\tilde{\psi}$ solves a similar equation as in (3.2) with wave speed $\omega_0 + \alpha_0$. We obtain

$$\mathcal{L}_{(\omega_0, A_0)} = \frac{\partial^4}{\partial x^4} - \frac{\partial^2}{\partial x^2} + \omega_0 - \psi = \frac{\partial^4}{\partial x^4} - \frac{\partial^2}{\partial x^2} + \omega_0 + \alpha_0 - \tilde{\psi} =: \mathcal{L}_{(\omega_0 + \alpha_0, \tilde{A}_0)}.$$

Last equality gives us the desired result.

Next, we can write the solution (3.1) as a Fourier series of the form (see [17])

$$\phi(x) = a_0 + \gamma \sum_{n=1}^{\infty} n \operatorname{csch} \left(\frac{n\pi K'_0}{K_0} \right) \cos \left(\frac{2\pi n}{L_0} x \right),$$

where $\gamma_0 := \left(\frac{b_0\pi^2}{K_0^2} + \frac{d_0\pi^2}{k^2 K_0^2} \left(\frac{4-2k_0^2}{3} + \frac{n^2\pi^2}{6K_0} \right) \right)$, $K_0 = K(k_0)$ and $K'(k_0) = K(\sqrt{1-k_0^2})$. In addition, according with (3.3) and (3.4) we can write $a_0 = a(k_0, L_0, \omega_0)$, $b_0 = b(k_0, L_0)$, $d_0 = d(k_0, L_0)$. Therefore, the Fourier coefficients are

$$\hat{\psi}(n) = \begin{cases} a_0, & n = 0 \\ \sigma(n), & n \neq 0 \end{cases} \quad (3.8)$$

where $\sigma(n) = \frac{\gamma_0}{2} n \operatorname{csch} \left(\frac{n\pi K'_0}{K_0} \right)$.

By defining $g(x) = \frac{\gamma_0}{2} x \operatorname{csch} \left(\frac{x\pi K'_0}{K_0} \right)$, $x \in \mathbb{R}$, we see that g is a smooth logarithmically concave function. Thus, from Lemma 4.1 in [2] one has that g belongs to the $PF(2)$ -continuous class and thus, since $a_0 > g(0)$, for all $\omega_0 > 0$ large enough, we can redefine the $PF(2)$ -continuous function g by a differentiable function $s : \mathbb{R} \rightarrow \mathbb{R}$ such that $s(0) = a_0$, $s(x) = g(x)$ in $(-\infty, -1] \cup [1, +\infty)$ such that $s \in PF(2)$ in the continuous case. Letting $s(n) = \hat{\psi}(n)$, $n \in \mathbb{Z}$, one has that $(\hat{\psi}(n))_{n \in \mathbb{Z}}$ belongs to $PF(2)$, for all $\omega_0 > 0$ large enough. Therefore, Theorem 4.1 in [2] gives us the spectral properties required in assumption (H). \square

Next, we need to analyze the difference $\frac{M(\psi)}{L_0} - \omega_0$ to conclude the orbital stability of periodic waves for the model (1.7). Indeed, since $M(\psi) = a_0 L$, we can deduce from (3.3) that $a_0 - \omega_0$ just depends on the pair $(k_0, L_0) \in J \times I$. Therefore, one has

$$\begin{aligned} \frac{M(\psi)}{L_0} - \omega_0 &= a_0 - \omega_0 \\ &= \frac{302848(-k_0^4 + k_0^2 + 1)K_0^4 + 14560L^2K_0^2(k_0^2 - 2) + 43680L^2E_0K_0 - 31L^4}{507L^4} \\ &= \frac{1}{507L^4} p(k_0, L_0^2), \end{aligned}$$

where $E_0 = E(k_0)$. By taking $L_1 = L_0^2$ we can rewrite function $p(k_0, L_0^2)$ as $p(k_0, L_1)$. This change of variables can be used to simplify the implicit relation in k_0 and L_0 in (3.6) as

$$\frac{89989120}{31}(k_0^2 - 2) \left(k_0^2 - \frac{1}{2}\right) (k_0^2 + 1)K_0^6 - \frac{908544}{31}L_1(k_0^4 - k_0^2 + 1)K_0^4 + L_1^3 = 0. \quad (3.9)$$

Using *Maple 16*, we can solve algebraically the equation in (3.9) in terms of the modulus in order of obtaining the positive function

$$L_1(k_0) = \frac{104}{31} \frac{r(k_0)}{q(k_0)},$$

where $r(k_0)$ and $q(k_0)$ are complicated expressions containing several powers of k_0 . Since $\frac{M(\psi)}{L_0} - \omega_0 = \frac{1}{507L^4}p(L_1(k_0))$, we can plot the graph of $p(k_0, L_1) := p(L_1(k_0))$ in order to understand its behaviour in terms of the modulus. The figure below shows that there are values of the pair k_0 such that the difference $p(L_1(k_0))$ is positive as required in our stability approach. Important to mention that our results are

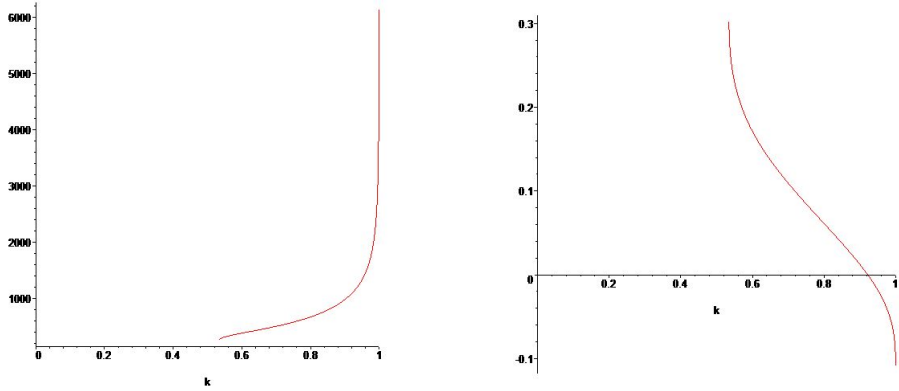


FIGURE 3.1. Left: The graph of the function $L_1(k_0)$. Right: The graph of $p(L_1(k_0))$.

agreeing with those ones in [7] since to conclude the stability in refereed paper, it makes necessary to analyse the behaviour of the difference $\frac{M(\psi)}{L_0} - \omega_0$. The main problem in [7] is that we need, in order to use an adaptation of the arguments in [10], to consider small values of $\omega_0 > 0$ to determine a positiveness of a certain quantity. This fact is not necessary and our stability result becomes more complete. Thus, collecting all results above we are enable to enunciate the following result.

Theorem 3.1. *Consider $L_0 > 0$ satisfying (3.6) and let $\omega_0 > 0$ be arbitrary but fixed. The traveling wave $\psi(x - \omega_0 t)$ in (3.1) is orbitally stable in $H_{per}^2([0, L_0])$ by the periodic flow of the equation (1.7) provided that $\frac{M(\psi)}{L_0} > \omega_0$.*

Remark 3.1. *Global solutions in the energy space $H_{per}^2([0, L_0])$ as well as existence of convenient conserved quantities as in (2.4), (2.5) and (2.5) with $\mathcal{M} = \partial_x^4 - \partial_x^2$ associated with the equation (1.7) can be found in reference [15].*

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REFERENCES

- [1] T.P. Andrade and A. Pastor, *Orbital stability of periodic traveling-wave solutions for the BBM equation with fractional nonlinear term*, to appear in Physica D, (2016).
- [2] J. Angulo and F. Natali, *Positivity properties of the Fourier transform and the stability of periodic travelling-wave solutions*, SIAM J. Math. Anal., 40 (2008), pp. 1123-1151.
- [3] J. Angulo, J. L. Bona and M. Scialom, *Stability of cnoidal waves*, Advances in Differential Equations, 11 (2006), pp. 1321-1374.
- [4] T. B. Benjamin, *The stability of solitary waves*, Proc. Roy. Soc. (London) Ser. A 328 (1972), pp. 153-183.
- [5] J.L. Bona, P.E. Souganidis and W.A. Strauss, *Stability and instability of solitary waves of Korteweg-de Vries type*, Proc. Roy. Soc. London Ser. A 411 (1987), pp. 395-412.
- [6] J. L. Bona, *On the stability theory of solitary waves*, Proc. R. Soc. Lond. Ser. A, 344(1975), pp. 363-374.
- [7] F. Cristófani, F. Natali and T.P. Andrade, *Orbital stability of periodic traveling wave solutions for the Kawahara equation*, preprint (2016).
- [8] F. Cristófani, F. Natali and A. Pastor, *Orbital stability of periodic traveling-wave solutions for the Log-KdV equation*, preprint (2016). <http://arxiv.org/pdf/1408.1709.pdf>
- [9] R.L. Frank and E. Lenzmann, *Uniqueness of non-linear ground states for fractional Laplacians in \mathbb{R}* , Acta Math. 210 (2013), pp. 261318.
- [10] M. Grillakis, J. Shatah and W. Strauss, *Stability theory of solitary waves in the presence of symmetry I*, J. Funct. Anal. 74 (1987), pp. 160-197.
- [11] M. Hărăguș and T. Kapitula, *On the spectra of periodic waves for infinite-dimensional Hamiltonian systems*, Phys. D 237 (2008), pp. 2649-2671.
- [12] M. Hărăguș, E. Lombardi and A. Scheel, *Spectral stability of wave trains in the Kawahara equation*, J. Math. Fluid Mech., 8 (2006), pp. 482-509.
- [13] V.M. Hur and M. Johnson, *Stability of periodic traveling waves for nonlinear dispersive equations*, SIAM J. Math. Anal., 47, pp. 35283554.
- [14] M. Johnson, *Nonlinear stability of periodic traveling wave solutions of the generalized Korteweg-de Vries equation*, SIAM J. Math. Anal., 41 (2009), pp. 1921-1947.
- [15] T. Kato, *Low regularity well-posedness for the periodic Kawahara equation*, Diff. Int. Equat., 25 (2012), pp. 1011-1036.
- [16] T. Kato, *Perturbation theory for linear Operators*, Springer, Berlin, (1976).
- [17] A. Kiper, *Fourier series coefficients for powers of the Jacobian Elliptic Functions*, Math. Comput., 43 (1984), pp. 247-259.
- [18] F. Natali and A. Neves, *Orbital stability of solitary waves*. IMA Journal of Applied Mathematics, 79 (2014), pp. 1161-1179.
- [19] E.J. Parkes, B.R. Duffy and P.C. Abbot, *The Jacobi elliptic-function method for finding periodic-wave solutions to nonlinear evolution equations*. Phys.Lett. A, 295 (2002), pp. 280-286.
- [20] M.I. Weinstein, *Modulation Stability of Ground States of Nonlinear Schrödinger Equations*. SIAM J. Math., 16 (1985), pp. 472-490.
- [21] M.I. Weinstein, *Liapunov stability of ground states of nonlinear dispersive equations*. Comm. Pure Appl. Math., 39 (1986), pp. 51-68.